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Measurement-Induced Nonlocality in an n-Partite Quantum State

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Abstract

We generalize the concept of measurement-induced non-locality (MIN) to an n-partite quantum state. We get exact analytical expressions for MIN in an n-partite pure and n-qubit mixed state. We obtain the conditions under which MiN equals geometric quantum discord in an n-partite pure state and an n-qubit mixed state.

Keywords: Measurement, Non-Locality, Quantum Discord, Quantum Information.

ملخص: تم تعميم مفهوم عدم المحلية المستحثة بالقياس الي حالة كمية من ن ـ جزاء. تم الحصول على تعبير تحليلي دقيق لعدم المحلية المستحثة بالقياس في حالة نقية وحالة خليطة من ن ـ كيوبت. اوجدنا الشروط التي عندها عدم المحلية المستحثة بالقياس تساوي الدسكورد الكمي الهندسي في حالة نقية من ن ـ جزاء وحالة خليطة من ن ـ كيوبت.

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1 Introduction

Quantum correlation, a fundamental aspect of quantum mechanics, significantly makes the departure from the classical regime. It is a useful physical resource for various types of quantum information processing, such as teleportation, super dense coding, communication and quantum algorithms [1]. Recently, various measures have been proposed to capture quantumness which go beyond entanglement such as measurement-induced disturbance (MID) [2], geometric discord (GD) [3, 4], measurement-induced nonlocality (MIN) [5] and uncertainty-induced nonlocality (UIN)[6]. The measurement induced non-locality (MIN) is a measure of quantum correlations as manifested in the non-local effects of local (on a single part) quantum operations [7]. These local quantum operations leave invariant the reduced density operators of the parts on which they act, while changing the global quantum state. MiN concerns the von-Neumann measurement on a part of a quantum system. Non-locality (in all its forms) being an inherently quantum phenomenon, is expected to be useful as a tool for quantitative specification

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of quantum correlation. Such a quantitative specification of quantum correlations in terms of MIN was given in [7] for bipartite quantum systems. MIN has also been investigated based on relative entropy [8], von Neumann entropy, skew information [9], trace distance [10, 11], fidelity [12] and MIN based on affinity [13]. Here we generalize this measure to N-partite quantum systems. The MIN is a manifestation of the quantum verses classical paradigm of quantum correlations and naturally compares with quantum discord [14] which is also a manifestation of such a paradigm. In fact, it is quite relevant to inquire about the conditions on quantum states under which MIN and geometric discord are equal (or, rather are different) and the different kinds of information they give about the quantum correlations in a quantum state. Here we establish such general conditions in n-partite pure and n-qubit mixed states.

2 Multipartite Generalization of MiN

Multipartite generalization of the MIN can be obtained in a manner analogous to that of a geometric quantum discord [14]. For an n-partite system in a stete ρ we define

$$N_l(\rho) = \frac{d_l}{d_l - 1} \max_{\Pi^{(l)}} ||\rho - \Pi^{(l)}(\rho)||^2), l = 1, 2, \cdots, n \tag{1}$$

where $\Pi^{(l)} = \{\Pi_k^{(l)}\}$ stands for the set of von-Neumann measurements on the lth part such as that $\Pi^{(l)}(\rho^{(l)}) = \sum_k \Pi_k^{(l)} \rho^{(l)} \Pi_k^{(l)} = \rho^{(l)}$, $\rho^{(l)}$ being the reduced density operator obtained by tracing out all parts other than the lth part. Such a measurement $\Pi^{(l)}$ is defined by the projectors corresponding to the eigenstates of $\rho^{(l)}$. When all the eigenvalues of $\rho^{(l)}$ are non-degenerate, there is only one von-Neumann measurement $\Pi^{(l)}$ satisfying $\Pi^{(l)}(\rho^{(l)}) = \sum_k \Pi_k^{(l)} \rho^{(l)} \Pi_k^{(l)} = \rho^{(l)}$ and the maximization requirement in Eq.(1) drops out. If one or more eigenvalues of $\rho^{(l)}$ are degenerate, the right hand side of the Eq.(1) has to be maximized over the eigenspaces of degenerate eigenvalues, which is, in general, a difficult task. Comparing the definitions of MIN $N_l(\rho)$ and the geometric discord $D_l(\rho)$ it follows that, for any n-partite state, $N_l(\rho) \geq D_l(\rho)$. We are interested in finding the criteria for their equality. The multipartite non-locality can be evaluated for an n-partite pure state via the following
Theorem 1: Let $|\psi\rangle = \sum_{i_1 i_2 \cdots i_n} a_{i_1 i_2 \cdots i_n} |i_1 i_2 \cdots i_n\rangle$ be a n-partite pure state. Then

$$N_l(|\psi\rangle\langle\psi|) = \frac{d_l}{d_l - 1}(1 - tr(\rho^{(l)})^2) \tag{2}$$

where $\rho^{(l)}$ is the reduced density matrix of the lth part snd $d_l = dim(H^{(l)})$.
Proof: In order to get the $N_l(|\psi\rangle\langle\psi|)$ we can directly calculate the terms which define it (Eq.(1)). We have

$$\rho = |\psi\rangle\langle\psi| = \sum_{i_1 i_2 \cdots i_n} \sum_{j_1 j_2 \cdots j_n} a_{i_1 i_2 \cdots i_n} a^*_{j i_1 j_2 \cdots j_n} |i_1 i_2 \cdots i_n\rangle\langle j_1 j_2 \cdots j_n|$$

The set o von-Neumann measurements on the lth part is given by

$$\Pi^{(l)} = \{\Pi_k^{(l)} = U|k_l\rangle\langle k_l|U^\dagger\}$$

where the $\{|k_l\rangle\}$, $k_l = 1, \ldots, d_l = dim(H^{(l)})$ is an orthonormal basis in $H^{(l)}$ and U is a unitary operator acting on $H^{(l)}$. A direct calculation of $tr(\rho\Pi^{(l)}\rho)$ and comparison with $\rho^{(l)} = tr_{\bar{l}}(\rho)$

gives, assuming that $\{U|k_l\rangle\}$ is the eigenbasis of $\rho^{(l)}$, $tr(\rho\Pi^{(l)}(\rho)) = \sum_{k_l}(\langle k_l|U^\dagger\rho^{(l)}U|k_l\rangle)^2 = \sum_{k_l}\lambda_{k_l}^2 = tr(\rho^{(l)})^2$ where $\{\lambda_{k_l}\}$ are the eigenvalues of $\rho^{(l)}$. From the definition of $N_l(\rho)$ (Eq.(1)) we get

$$N_l(\rho) = \frac{d_l}{d_l-1}||\rho||^2 - min_{\Pi^{(l)}}(2tr(\rho\Pi^{(l)}(\rho)) - ||\Pi^{(l)}(\rho)||^2)$$

For a pure state $||\rho||^2 = 1$ and $||\Pi^{(l)}(\rho)||^2 = tr(\rho\Pi^{(l)}(\rho))$ so that

$$N_l(\rho) = \frac{d_l}{d_l-1}(1 - tr(\rho\Pi^{(l)}(\rho)))$$

The minimum is over the von-Neumann measurements leaving the marginal state $\rho^{(l)}$ invariant, that is $\sum_k \Pi_k^{(l)}\rho^{(l)}\Pi_k^{(l)} = \rho^{(l)}$, or,

$$\sum_{k_l}\langle k_l|U^\dagger\rho^{(l)}U|k_l\rangle U|k_l\rangle\langle k_l|U^\dagger = \rho^{(l)}$$

This is the spectral decomposition of the $\rho^{(l)}$ which is consistent with our choice of $\{U|k_l\rangle$ to be the eigenbasis of $\rho^{(l)}$. Since the $tr(\rho\Pi^{(l)}(\rho))$ is simply the trace of $(\rho^{(l)})^2$, the minimization in the definition of the N_l Eq.(1) drops out and we get

$$N_l(|\psi\rangle\langle\psi|) = \frac{d_l}{d_l-1}(1 - tr(\rho^{(l)})^2)$$

Corollary: For an n-partite pure state $\rho = |\psi\rangle\langle\psi|$

$$D_l(\rho) = N_l(\rho)$$

where the $D_l(\rho)$ is the geometric discord of the ρ with the von-Neomann measurment on the lth part. This important result follows trivially, because the $D_l(\rho)$ requires maximization over all von-Neumann measurements on the lth part which is obtained if the $\{U|k_l\rangle\}$ forms the eignbasis of $\rho^{(l)}$.

It is interesting to compare the $N_l(\rho)$ (Eq.(1) with measures of entanglement of pure multipartite states. For a bipartite pure state ρ_{AB} we have, for the concurrence,

$$C(\rho_{AB}) = \sqrt{2(1 - tr(\rho_A^2)}$$

which is related to the $N_l(\rho)_{AB}$ by the

$$N_l(\rho)_{AB} = \frac{d_l}{2(d_l-1)}C^2(\rho_{AB}).$$

Thus, for pure bipartite states, non-locality is simply related to concurrence.

The Meyer-Wallach measure of entanglement of multipartite pure states is

$$Q(|\psi\rangle) = \frac{1}{n}\sum_{k=1}^{n} 2(1 - tr(\rho_k^2))$$

where ρ_k is the reduced density operator for the kth part. Thus,

$$Q(|\psi\rangle) = \frac{2}{n}\sum_{l=1}^{n}\left(\frac{d_l-1}{d_l}\right)N_l(|\psi\rangle\langle\psi|).$$

Thus the Meyer-Wallach measure of a pure state multipartite entanglement is the average of a non-locality over the parts of the system.

3 Non-locality in the Multipartite Mixed States

To get an $N_l(\rho)$ in this case, we start with the Bloch representation of a multipartite state ρ[15]. Bloch representation [15] of an n-partite density matrix is

$$\rho = \frac{1}{\Pi_k^N d_k}\{\otimes_k^N I_{d_k} + \sum_{k\in\mathcal{N}}\sum_{\alpha_k} s_{\alpha_k}\lambda_{\alpha_k}^{(k)} + \sum_{2\leq M\leq N}\sum_{\{k_1,k_2,\cdots,k_M\}}\sum_{\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_M}} \tilde{t}_{\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_M}}\lambda_{\alpha_{k_1}}^{(k_1)}\lambda_{\alpha_{k_2}}^{(k_2)}\cdots\lambda_{\alpha_{k_M}}^{(k_M)}\}. \quad (3)$$

Where

$$\lambda_{\alpha_{k_1}}^{(k_1)} = (I_{d_1}\otimes I_{d_2}\otimes\cdots\otimes\lambda_{\alpha_{k_1}}\otimes I_{d_{k_1+1}}\otimes\cdots\otimes I_{d_N})$$

$$\lambda_{\alpha_{k_2}}^{(k_2)} = (I_{d_1}\otimes I_{d_2}\otimes\cdots\otimes\lambda_{\alpha_{k_2}}\otimes I_{d_{k_2+1}}\otimes\cdots\otimes I_{d_N})$$

$$\lambda_{\alpha_{k_1}}^{(k_1)}\lambda_{\alpha_{k_2}}^{(k_2)} = (I_{d_1}\otimes I_{d_2}\otimes\cdots\otimes\lambda_{\alpha_{k_1}}\otimes I_{d_{k_1+1}}\otimes\cdots\otimes\lambda_{\alpha_{k_2}}\otimes I_{d_{k_2+1}}\otimes I_{d_N}) \quad (4)$$

$\mathbf{s}^{(k)}$ is a Bloch vector corresponding to kth subsystem, $\mathbf{s}^{(k)} = [s_{\alpha_k}]_{\alpha_k=1}^{d_k^2-1}$ and

$$\tilde{t}_{\alpha_{k_1}\alpha_{k_2}\ldots\alpha_{k_M}} = \frac{d_{k_1}d_{k_2}\ldots d_{k_M}}{2^M}Tr[\rho\lambda_{\alpha_{k_1}}^{(k_1)}\lambda_{\alpha_{k_2}}^{(k_2)}\cdots\lambda_{\alpha_{k_M}}^{(k_M)}] \quad (5)$$

For more details see ref.[15, 16, 17]. We need to define a product of a tensor with a matrix, the n-mode product [18, 19]. The *n-mode (matrix) product* of a tensor $\mathcal{Y}$ (of order n and with dimension $J_1\times J_2\times\cdots J_N$) with a matrix A with dimension $I\times J_n$ is denoted by $\mathcal{Y}\times_n A$. The result is a tensor of size $J_1\times J_2\times\cdots J_{n-1}\times I\times J_{n+1}\times\cdots J_N$ and is defined elementwise by

$$(\mathcal{Y}\times_n A)_{j_1j_2\cdots j_{n-1}ij_{n+1}\cdots j_N} = \sum_{j_n=1}^{J_n} y_{j_1j_2\cdots j_N}a_{ij_n}. \quad (6)$$

Recently, for a bipartite system ab ($N=2$) with states in $\mathcal{H}^a\otimes\mathcal{H}^b$, $dim(\mathcal{H}^a) = d_a$, $dim(\mathcal{H}^b) = d_b$, S. Luo and S. Fu introduced the Measurement-Induced Nonlocality [7]

$$N_a(\rho) = tr(TT^t) - \min_A tr(ATT^tA^t), \quad (7)$$

where $T=[t_{ij}]$ is an $d_a^2\times d_b^2$ matrix and the minimum is taken over all the $d_a\times d_a^2-1$-dimensional isometric matrices $A=[a_{ji}]$ such that $a_{ji} = tr(|j\rangle\langle j|X_i) = \langle j|X_i|j\rangle$, $j=1,2,\ldots,d_a$; $i = 1,2,\ldots,d_a^2-1$ and $\{|j\rangle\}$ is any orthonormal basis in the $\mathcal{H}^a$. we generalize this result to an n-partite quantum states.

Theorem 2. Let the $\rho_{12\cdots N}$ be an n-partite state defined by Eq.(4), then

$$N_l(\rho) = C_l\{\sum_{1\leq M\leq N-1}\sum_{\{k_1,k_2,\cdots,k_M\}\subseteq\mathcal{N}-\{l\}}\frac{d_ld_{k_1}d_{k_2}\cdots d_{k_M}}{2^{M+1}}||\mathcal{T}^{\{l,k_1,k_2,\cdots,k_M\}}||^2 - \min_{A^{(l)}} tr(A^{(l)}K^{(l)}(A^{(l)})^t)\}, \quad (8)$$

where $C_l = \frac{d_l}{(d_l-1)\Pi_k^N d_k}$, $K^{(l)}$ define as

$$K_{\alpha_l\beta_l}^{(l)} = \sum_{1\leq M\leq N-1}\sum_{\{k_1,k_2,\cdots,k_M\}\subseteq\mathcal{N}-\{l\}}\sum_{\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_M}}\frac{d_ld_{k_1}d_{k_2}\cdots d_{k_M}}{2^{M+1}}t_{\alpha_l\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_M}}t_{\beta_l\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_M}},$$

and $\mathcal{T} = [t_{i_1 i_2 \cdots i_M}] = [Tr(\rho\lambda^{(k_1)}_{\alpha_{k_1}}\lambda^{(k_2)}_{\alpha_{k_2}}\cdots\lambda^{(k_M)}_{\alpha_{k_M}})]$, the maximum is taken over all $d_l \times (d_l^2 - 1)$-dimensional matrices $A^{(l)} = [a_{ji_l}]$, such that $a_{ji_l} = tr(|j\rangle\langle j|\frac{\lambda^{(l)}_{i_l}}{\sqrt{2}})$, $j = 1, 2, \ldots, d_l$; $i_l = 1, 2, \ldots, d_l^2 - 1$ and $\{|j\rangle\}$ is any orthonormal basis for $\mathcal{H}^{(l)}$. In particular, we have

$$N_l(\rho) \leq \sum_{i=1}^{d_l^2 - d_l} \eta_i, \tag{9}$$

where the $\{\eta_i : i = 1, 2, \cdots, d_l^2 - 1\}$ are the eigenvalues of the $(d_l^2 - 1) \times (d_l^2 - 1)$- symmetric matrix $K^{(l)}$ listed in a decreasing order. Furthermore, if the $\rho^{(l)} = tr_{\bar{l}}\,\rho_{12\cdots N}$ (where $tr_{\bar{l}}$ is taken trace over all parts except l^{th} part) is non-degenerate with spectral projections $\{|j\rangle\langle j|\}$, then

$$N_l(\rho) = C_l\{\sum_{1\leq M\leq N-1}\sum_{\{k_1,k_2,\cdots,k_M\}\subseteq\mathcal{N}-\{l\}} \frac{d_l d_{k_1} d_{k_2}\cdots d_{k_M}}{2^{M+1}}||\mathcal{T}^{\{l,k_1,k_2,\cdots,k_M\}}||^2 - tr(A^{(l)}K^{(l)}(A^{(l)})^t)\}, \tag{10}$$

Theorem 3. If the lth part is a qubit ($d_l = 2$), then

$$N_l(\rho) = C_l\left[\sum_{1\leq M\leq N-1}\sum_{\{k_1,\ldots,k_M\}\subseteq\mathcal{N}-\{l\}} \frac{d_{k_1} d_{k_2}\cdots d_{k_M}}{2^M}||\mathcal{T}^{\{l,k_1,\ldots,k_M\}}||^2 - \begin{cases} \frac{\mathbf{s}^{(l)t}K^{(l)}\mathbf{s}^{(l)}}{||\mathbf{s}^{(l)}||^2}, & \text{if } \mathbf{s}^{(l)} \neq 0 \\ \eta_{min}, & \text{if } \mathbf{s}^{(l)} = 0 \end{cases}\right], \tag{11a}$$

where the $\mathbf{s}^{(l)}$ is the coherent vector of $\rho^{(l)}$ and η_{min} is the smallest eigenvalue of the matrix $K^{(l)}$ which is an 3×3 real symmetric matrix, defined as

$$K^{(l)} = \sum_{1\leq M\leq N-1}\sum_{\{k_1,\ldots,k_M\}\subseteq\mathcal{N}-\{l\}}\sum_{\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_M}} \frac{d_{k_1} d_{k_2}\cdots d_{k_M}}{2^M} t_{\alpha_l\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_M}} t_{\beta_l\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_M}}, \tag{11}$$

For n-qubit ($d_i = 2,\ i = 1, 2, \cdots, n$), then

$$N_l(\rho) = \frac{1}{2^{(N-1)}}\left[\sum_{1\leq M\leq N-1}\sum_{\{k_1,\ldots,k_M\}\subseteq\mathcal{N}-\{l\}} ||\mathcal{T}^{\{l,k_1,\ldots,k_M\}}||^2 - \begin{cases} \frac{\mathbf{s}^{(l)t}K^{(l)}\mathbf{s}^{(l)}}{||\mathbf{s}^{(l)}||^2}, & \text{if } \mathbf{s}^{(l)} \neq 0 \\ \eta_{min}, & \text{if } \mathbf{s}^{(l)} = 0 \end{cases}\right] \tag{12a}$$

$$K^{(l)} = \sum_{1\leq M\leq N-1}\sum_{\{k_1,\ldots,k_M\}\subseteq\mathcal{N}-\{l\}}\sum_{\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_M}} t_{\alpha_l\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_M}} t_{\beta_l\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_M}}, \tag{12}$$

The proof of the theorems (2) and (3) is a straight forward generalization of that of theorems (2) and (3) respectively in Ref. [7] to the multipartite case, so that we skip it.

4 The Relation Between the Non-Locality and the Geometric Quantum Discord for Arbitrary n-qubit States

We saw (see(Eq.(2))) that the non-locality and geometric discord are equal for the arbitrary n-partite pure states. In this section we find a class of the n-qubit states for which these quantities coincide. Consider an n-qubit state ρ. The geometric discord for such a state corresponding to the von-Neumann measurement on lth qubit is given by

$$D_l(\rho) = \frac{1}{2^{(N-1)}}\left[||s^{(l)}||^2 + \sum_{1\leq M\leq N-1}\sum_{\{k_1,\ldots,k_M\}\subseteq\mathcal{N}-\{l\}} ||\mathcal{T}^{\{l,k_1,\ldots,k_M\}}||^2 - \lambda_{max}\right], \tag{13}$$

where $s^{(l)}$ is the coherent vector and $\rho^{(l)}$ (reduced density operator of the lth part), $\mathcal{T} = [t_{\alpha_{k_1}\alpha_{k_2}\dots\alpha_{k_M}}] = [Tr(\rho\lambda^{(k_1)}_{\alpha_{k_1}}\lambda^{(k_2)}_{\alpha_{k_2}}\cdots\lambda^{(k_M)}_{\alpha_{k_M}})]$, and λ_{max} is the largest eigenvalue of the 3×3 real symmetric matrix

$$G^{(l)} = \mathbf{s}^{(l)}(\mathbf{s}^{(l)})^t + K^{(l)} \tag{14}$$

where $K^{(l)}$ is given by Eq.(12) for the n-qubits. The non-locality for the n-qubit state ρ is given by Eq.(11a). We now consider two cases.
Case I: $\mathbf{s}^{(l)} \neq 0$. By Eq.(14) we get the $\hat{e}^t G^{(l)}\hat{e} = \hat{e}^t\mathbf{s}^{(l)}\mathbf{s}^{(l)}\hat{e} + \hat{e}^t K^{(l)}\hat{e}$ where $\hat{e}\in R^3$ is an arbitrary unit vector, choosing $\hat{e} = \frac{\mathbf{s}^{(l)}}{||\mathbf{s}^{(l)}||}$, we get

$$\frac{(\mathbf{s}^{(l)})^t K^{(l)}\mathbf{s}^{(l)}}{||\mathbf{s}^{(l)}||^2} = \frac{(\mathbf{s}^{(l)})^t G^{(l)}\mathbf{s}^{(l)}}{||\mathbf{s}^{(l)}||^2} - ||\mathbf{s}^{(l)}||^2$$

Substituting in Eq.(11) we get

$$N_l(\rho) = \frac{1}{2^{(N-1)}}\left[||\mathbf{s}^{(l)}||^2 + \sum_{1\le M\le N-1}\ \sum_{\{k_1,\dots,k_M\}\subseteq\mathcal{N}-\{l\}} ||\mathcal{T}^{\{l,k_1,\dots,k_M\}}||^2 - \frac{\mathbf{s}^{(l)^t}G^{(l)}\mathbf{s}^{(l)}}{||\mathbf{s}^{(l)}||^2}\right] \tag{15}$$

If the $\frac{\mathbf{s}^{(l)}}{||\mathbf{s}^{(l)}||}$, is the eigenvector of the $G^{(l)}$ with the largest eigenvalue then the right hand side of the Eq.(15) gives the geometric discord $D_l(\rho)$ so that under this condition $N_l(\rho) = D_l(\rho)$. The above condition can be equivalently stated as $[\mathbf{s}^{(l)}(\mathbf{s}^{(l)})^t, K(l)] = 0$ and $||\mathbf{s}^{(l)}||^2 + \eta_l \geq \eta_{i\neq l}$ where $\{\eta_i\}$ are the eigenvalues of $K^{(l)}$ and the η_l is the eigenvalue corresponding to the eigenvector $\frac{\mathbf{s}^{(l)}}{||\mathbf{s}^{(l)}||}$.
Case II: $\mathbf{s}^{(l)} = 0$. In this case the ρ has one doubly degenerate eigenvalue. With $\mathbf{s}^{(l)} = 0$ we get

$$\hat{e}^t G^{(l)}\hat{e} = \hat{e}^t K^{(l)}\hat{e}. \tag{16}$$

To get a non-locality we have to minimize the right hand side while the geometric discord requires maximization of the left hand side. Under these conditions, the equality in Eq.(16) is preserved if the $G^{(l)} = K^{(l)}$ has a single three-fold degenerate eigenvalue,($\eta_1 = \eta_2 = \eta_3$). Thus when $\mathbf{s}^{(l)} = 0$, $N_l(\rho) = D_l(\rho)$ provided the matrix $K^{(l)}$ has a single three fold degenerate eigenvalue.

As a first example we consider the set of three qubit states comprising the convex combination of the $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and $|w\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$,

$$\rho(p) = p|GHZ\rangle\langle GHZ| + (1-p)|w\rangle\langle w|$$

the $K^{(1)}$ matrix of this state is

$$K^{(1)} = \begin{bmatrix} 2p^2 + \frac{16}{9}(1-p)^2 & 0 & 0 \\ 0 & 2p^2 + \frac{16}{9}(1-p)^2 & 0 \\ 0 & 0 & 2p^2 + \frac{19}{9}(1-p)^2 - \frac{4}{3}p(1-p) \end{bmatrix},$$

with the coherent vector for the first qubit $\mathbf{s}^{(1)} = [0\ 0\ \frac{1}{3}(1-p)]^t \neq 0$. So that case I applies. The $[\mathbf{s}^{(1)}(\mathbf{s}^{(1)})^t, K(1)] = 0$, and the condition $||\mathbf{s}^{(1)}||^2 + \eta_1 \geq \eta_{i\neq 1}$, η_1 is the eigenvalue of $K^{(1)}$ matrix corresponding to eigenvector $\frac{\mathbf{s}^{(1)}}{||\mathbf{s}^{(1)}||}$, is satisfied when $p \leq \frac{1}{4}$ and $p = 1$. This is depicted in Fig.(1a). The second example consists of the

$$\rho(p) = p|\tilde{w}\rangle\langle\tilde{w}| + (1-p)|w\rangle\langle w|$$

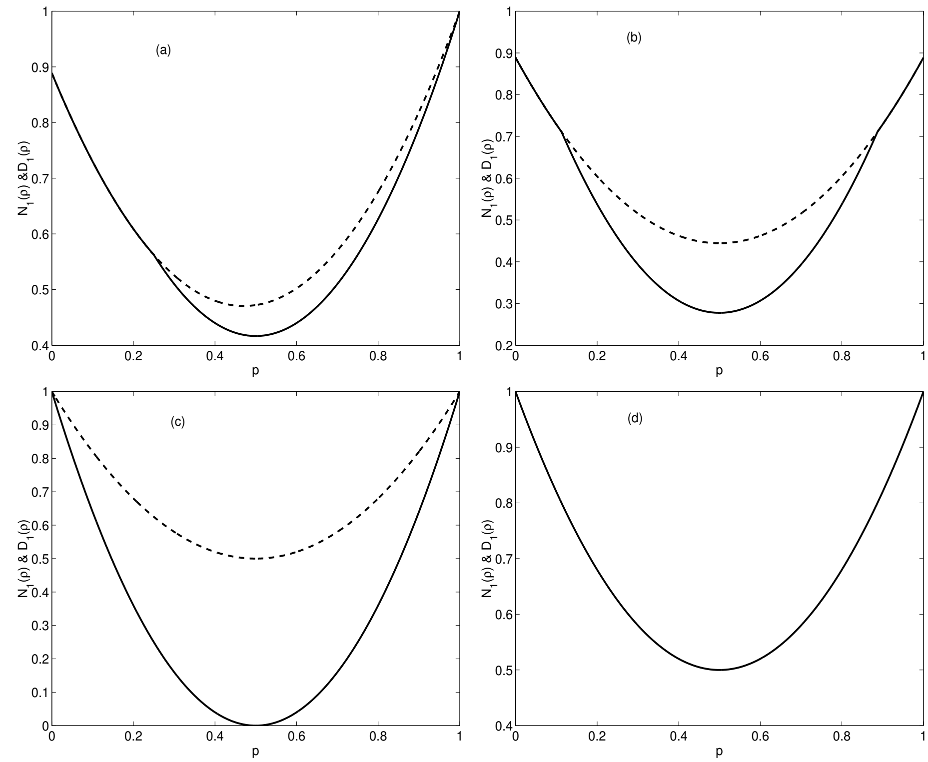

Figure 1: MIN (dashed line) and GQD (solid line) for different Mixed states of three qubits

where $|\tilde{w}\rangle$ is the flipped $|w\rangle$ state, $\sigma_x \otimes \sigma_x \otimes \sigma_x|w\rangle$. The $K^{(1)}$ matrix of this state is

$$K^{(1)} = \begin{bmatrix} \frac{16}{9}p^2 + \frac{16}{9}(1-p)^2 & 0 & 0 \\ 0 & \frac{16}{9}p^2 + \frac{16}{9}(1-p)^2 & 0 \\ 0 & 0 & \frac{19}{9}p^2 + \frac{19}{9}(1-p)^2 - \frac{10}{3}p(1-p) \end{bmatrix},$$

with the coherent vector for the first qubit $\mathbf{s}^{(1)} = [0\ 0\ \frac{1}{3}(1-2p)]^t \neq 0$. So that case I applies. The $[\mathbf{s}^{(1)}(\mathbf{s}^{(1)})^t, K(1)] = 0$ and the condition $||\mathbf{s}^{(1)}||^2 + \eta_1 \geq \eta_{i\neq 1}$, η_1 is satisfied when the $p \leq 0.1127$ and $p \geq 0.8873$. The results are shown in Fig.(1b). The third example consists of

$$\rho(p) = p|GHZ\rangle\langle GHZ| + (1-p)|GHZ_-\rangle\langle GHZ_-|$$

where the $|GHZ_-\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$. The $K^{(1)}$ matrix of this state is

$$K^{(1)} = \begin{bmatrix} 2(2p^2-1)^2 & 0 & 0 \\ 0 & 2(2p^2-1)^2 & 0 \\ 0 & 0 & 2(2p^2-1)^2 \end{bmatrix},$$

and the coherent vector for the first qubit $\mathbf{s}^{(1)} = 0$. So that case II applies. The $K^{(1)}$ does not have a single triply degenerate eigenvalue, for all the p, except for the $p = 0$ and $p = 1$. Therefore, $N_l(\rho) \neq D_l(\rho)$ for all the p between 0 and 1. The results are shown in Fig.(1c). The last example consists of the states

$$\rho(p) = p|GHZ\rangle\langle GHZ| + (1-p)|GHZ_1\rangle\langle GHZ_1|,$$

where $|GHZ_{-}\rangle = \frac{1}{\sqrt{2}}(|001\rangle - |110\rangle)$ The $K^{(1)}$ matrix of this state is

$$K^{(1)} = \begin{bmatrix} 2(p^2-(1-p)^2) & 0 & 0 \\ 0 & 2(p^2-(1-p)^2) & 0 \\ 0 & 0 & 2(p^2-(1-p)^2) \end{bmatrix},$$

and the coherent vector for the first qubit $\mathbf{s}^{(1)} = 0$. So that case II applies. The $K^{(1)}$ does have a single triply degenerate eigenvalue, for all the p. Therefore $N_l(\rho) = D_l(\rho)$ for all p as shown in Fig.(1d).

5 conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest. This research does not have financial support.

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